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Analytical solution for the transient temperature distribution in a moving rod or plate of finite length with surface heat transfer

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Abstract—An analytical solution for the transient temperature distribution in a wide flat plate and in a cylindrical rod of finite length moving at a constant speed and subjected to convective heat transfer at the surface is obtained. The analytical solution is obtained as an infinite series. However, inclusion of the first 25 terms of the series was found to be sufficient to obtain a converged solution in most cases. The analytical solutions are compared with previously obtained numerical solutions for this moving boundary problem. Excellent agreement between the analytical and numerical results is obtained, indicating the importance of the analytical solution for the validation of numerical schemes. The variation of the temperature field within the material with time is investigated. Even though this is one of the first attempts to solve, analytically, the problem of a finite-length moving material subjected to surface heat transfer, the analytical solutions are found to be of limited use at moderate or large times. However, at very small times, following the start of the process, numerical solutions are often quiet inaccurate. Then the analytical results are particularly useful. The versatility and ease of application of the numerical method are discussed.

1. INTRODUCTION

A VERY common circumstance encountered in several manufacturing processes is that of a moving plate or cylinder subjected to heat transfer at the surface. In the extrusion of metals, plastics or food materials, for instance, the material emerges from the extruder and cools by means of convection and radiation as it moves away from the die [1, 2]. Similar situations arise in hot rolling, wire drawing, glass fibre drawing, and continuous casting [3, 4]. In most cases, the time-dependent thermal field in the material, following the onset of the process, is of interest. As the length of the fed material increases the process generally reaches a 'pseudo-steady' state condition [5]. In several extrusion and crystal growth processes, however, the process may not last long enough to reach steady state and the transient problem is of particular importance.

A sketch of the process under consideration is shown in Fig. 1. At time $\tau' = 0$, the length of the cylinder or plate is zero and after a finite time τ', the instantaneous length, l, of the cylinder or plate moving at a constant speed of U_s is given by $l(\tau') = U_s\tau'$. The increment in the length over a finite time interval $\Delta\tau'$ is then $U_s\Delta\tau'$. Figure 1 shows the moving boundary of the material at three time intervals. Jaluria and Singh [5] have studied this problem numerically with an assumed heat transfer coefficient, h, at the surface. They employed finite-difference methods to discretize the governing equations, and solved the problem, using the Gauss–Seidel iterative procedure [6]. Analytical solutions for the steady state problem of a stationary cylinder or plate of given length, subjected to appropriate boundary conditions at the ends and the surface, are available in literature [7]. Some solutions are also available for the steady state case of an infinite moving cylinder or plate with lumping in radial/transverse direction [8]. However, there is no analytical solution available for the transient case of a moving material of finite length, subjected to surface energy transfer. The moving boundary considerably complicates this case. The motivation of the present work is to obtain an analytical solution to this transient problem. With the analytical solution in hand, different numerical schemes may be validated and checked for accuracy. The numerical approach is based on the assumption of unchanged length over a finite time step, $\Delta\tau'$. The results are not expected to be very accurate at small times, unless the time step is taken as extremely small. On the other hand, the analytical result, obtained as a series solution, is the exact solution and is thus correct at all times, provided an adequate number of terms in the series is employed.

The present work considers the heat transfer from a finite-length moving cylindrical rod, or a flat plate. The governing parameters are the cylinder speed and heat transfer coefficients from the end of the cylinder/plate and from the side. These physical quantities lead to two main dimensionless variables: the Peclet number, Pe, and the Biot number, Bi, defined later in the paper. For the feasibility of the analytical solution the Biot numbers are assumed to be known, that is the conjugate fluid flow problem is not considered and only conduction in the solid is investigated. When

NOMENCLATURE

Bi	surface Biot number, hR_0/k
Bi_L	end Biot number, h_1R_0/k
h	convective heat transfer coefficient from the surface
h_1	convective heat transfer coefficient from the end
J_0	Bessel function of first kind and of order 0
J_1	Bessel function of first kind and of order 1
k	thermal conductivity
l	instantaneous length, $U_s\tau'$
L	dimensionless instantaneous length, l/R_0
Pe	Peclet number, $U_s r_0/\alpha$
r	radial coordinate distance for cylinder
R	dimensionless radial coordinate distance for cylinder, r/R_0
R_0	radius of the cylinder or half-width of the plate
t	dimensionless transformed time, τ
U_s	cylinder or plate speed
y	transverse coordinate distance from flat plate centerline
Y	dimensionless transverse coordinate distance from flat plate centerline, y/R_0
z	axial coordinate distance
Z	dimensionless axial coordinate distance, z/R_0.

Greek symbols

α	thermal diffusivity
θ	dimensionless temperature, $\frac{(T-T_\infty)}{(T_o-T_\infty)}$
ξ	dimensionless transformed axial coordinate, $Z-(Pe)\tau$
τ'	physical time
τ	dimensionless time, $\alpha\tau'/R_0^2$
ϕ	'pseudo-steady' part of the transient solution
ψ	'pseudo-transient' part of the transient solution.

Subscripts

∞	ambient medium
o	base of cylindrical rod or flat plate.

the Biot number is small, the cylinder/plate can be considered as lumped in the transverse direction, giving rise to a one-dimensional problem. The solution to this simpler circumstance is also considered here.

The results obtained by the analytical solution are compared with the numerical results obtained by Jaluria and Singh [5]. A very close agreement is obtained at moderate or larger times. However, large differences arise at small times following the start of the process. The analytical results are found to be valuable in providing accurate results at short times, indicating quantitatively the basic trends in the transient transport and in providing a means for validating and developing appropriate numerical schemes. The case of implementing the numerical solution and its versatility in a wide range of practical problems are also discussed.

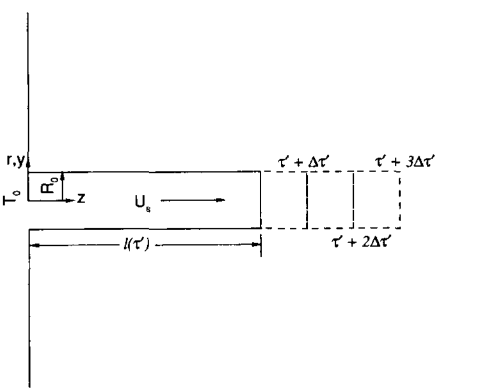


FIG. 1. Schematic diagram of the transient process for a material moving continuously.

2. ANALYTICAL SOLUTION FOR THE 2D TRANSIENT PROBLEM

2.1. *Cylindrical rod*

The governing equation for transient conduction in a cylindrical rod moving at a speed U_s may be written for constant material properties as:

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial \tau'}+U_s\frac{\partial T}{\partial z}\right)=\frac{\partial^2 T}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \quad (1)$$

The following non-dimensionalizations are employed to simplify and generalize the governing equations:

$$Bi=\frac{hR_0}{k},\quad Bi_L=\frac{h_1R_0}{k},\quad Z=z/R_0,\quad L=l/R_0,$$

$$R=r/R_0,\quad \theta=\frac{T-T_\infty}{T_o-T_\infty},\quad \tau=\frac{\alpha\tau'}{R_0^2},\quad Pe=\frac{U_sR_0}{\alpha} \quad (2)$$

where R_0 is the radius of the cylinder, T_o is the temperature of the base of the cylinder, T_∞ is the ambient temperature, and h and h_1 are the convective heat transfer coefficients at the surface of the cylinder and at the end, respectively. The instantaneous length l of the cylinder after time τ' is given by $l=U_s\tau'$. With this non-dimensionalization the governing equation becomes:

$$\frac{\partial \theta}{\partial \tau}+Pe\frac{\partial \theta}{\partial Z}=\frac{\partial^2 \theta}{\partial Z^2}+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right). \quad (3)$$

The following transformations may be employed to simplify the equation:

$$\xi = Z-(Pe)\tau = Z-L; \quad t=\tau. \quad (4)$$

Then equation (3) becomes:

$$\frac{\partial \theta}{\partial t}=\frac{\partial^2 \theta}{\partial \xi^2}+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right). \quad (5)$$

The initial and boundary conditions may be written in terms of these transformed non-dimensional variables as

$$\theta(\xi,R,0)=1.0$$

$$\theta(-L,R,t)=1.0;\quad \frac{\partial \theta}{\partial \xi}(0,R,t)=-Bi_{\mathrm{L}}\theta(0,R,t)$$

$$\frac{\partial \theta}{\partial R}(\xi,0,t)=0.0;\quad \frac{\partial \theta}{\partial R}(\xi,1,t)=-Bi\theta(\xi,1,t) \quad (6)$$

where Bi and Bi_{L} are defined in equation (2).

Since the boundary condition at $\xi=-L$ is non-homogeneous, it must be removed to make the problem admissible to a solution based on the separation of variables. Therefore, the principle of superposition may be used. Noting that the problem has a 'pseudo-steady' solution (that is in the transformed system, (ξ,R,t) the problem has a solution, that is independent of t, even though this solution depends on ξ which is a function of real time, τ; hence the name 'pseudo-steady') as $t\to\infty$, the following superposition is employed:

$$\theta(\xi,R,t)=\psi(\xi,R,t)+\phi(\xi,R) \quad (7)$$

such that $\phi(\xi,R)$ satisfies the following equation:

$$\frac{\partial^2 \phi}{\partial \xi^2}+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right)=0 \quad (8)$$

with the following boundary conditions:

$$\phi(-L,R)=1.0;\quad \frac{\partial \phi}{\partial \xi}(0,R)=-Bi_{\mathrm{L}}\phi(0,R)$$

$$\frac{\partial \phi}{\partial R}(\xi,0)=0.0;\quad \frac{\partial \phi}{\partial R}(\xi,1)=-Bi\phi(\xi,1). \quad (9)$$

That is, ϕ is the 'pseudo-steady' solution of the equation (5) with the boundary conditions given by equation (6). Therefore, ψ must satisfy the following equation:

$$\frac{\partial \psi}{\partial t}=\frac{\partial^2 \psi}{\partial \xi^2}+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \psi}{\partial R}\right) \quad (10)$$

with the following initial and boundary conditions:

$$\psi(\xi,R,0)=1.0-\phi(\xi,R)$$

$$\psi(-L,R,t)=0.0;\quad \frac{\partial \psi}{\partial \xi}(0,R,t)=-Bi_{\mathrm{L}}\psi(0,R,t)$$

$$\frac{\partial \psi}{\partial R}(\xi,0,t)=0.0;\quad \frac{\partial \psi}{\partial R}(\xi,1,t)=-Bi\psi(\xi,1,t). \quad (11)$$

The solution of equation (8) with the boundary conditions given by equation (9) can be obtained by the separation of variables. Assuming $\phi(\xi,R)=Z_1(\xi)R_1(R)$, the following sets of ordinary differential equations (ODEs) are obtained:

$$\frac{1}{R_1}\left[\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}R_1}{\mathrm{d}R}\right)\right]=-\frac{1}{Z_1}\frac{\mathrm{d}^2Z_1}{\mathrm{d}\xi^2}=-\lambda^2$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}R_1}{\mathrm{d}R}\right)+\lambda^2R_1=0 \quad (12)$$

$$\frac{\mathrm{d}^2Z_1}{\mathrm{d}\xi^2}-\lambda^2Z_1=0 \quad (13)$$

with

$$\frac{\mathrm{d}R_1}{\mathrm{d}R}(0)=0.0;$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}R}(1)=-BiR_1(1);$$

$$\frac{\mathrm{d}Z_1}{\mathrm{d}\xi}(0)=-Bi_{\mathrm{L}}Z_1(0). \quad (14)$$

The solution to equation (12) in conjunction with the boundary conditions, equation (14), gives:

$$R_{1n}(R)=A_nJ_0(\lambda_nR) \quad (15)$$

J_0 being the Bessel function of the first kind and of order 0. From the boundary condition, equation (14), $J_0'(\lambda_nR)(R=1)=-BiJ_0(\lambda_nR)(R=1)$. Using the properties of Bessel functions, the following equation may be written from which the roots λ_n can be obtained:

$$\lambda_nJ_1(\lambda_n)-BiJ_0(\lambda_n)=0. \quad (16)$$

From equation (13), the solution is obtained as:

$$Z_{1n}(\xi)=C_{1n}\mathrm{e}^{\lambda_n\xi}+C_{2n}\mathrm{e}^{-\lambda_n\xi}.$$

Using the boundary condition from equation (14) C_{2n} is eliminated to give

$$Z_{1n}(\xi)=C_n\left[\mathrm{e}^{\lambda_n\xi}+\frac{\lambda_n+Bi_{\mathrm{L}}}{\lambda_n-Bi_{\mathrm{L}}}\mathrm{e}^{-\lambda_n\xi}\right]. \quad (17)$$

Therefore, using the product solution, $\phi(\xi,R)$ can be written as:

$$\phi = \sum_{n=1}^{\infty} a_n \left[e^{\lambda_n \xi} + \frac{\lambda_n + Bi_L}{\lambda_n - Bi_L} e^{-\lambda_n \xi} \right] J_0(\lambda_n R) \quad (18)$$

where $a_n = A_n C_n$. But $\phi(-L,R) = 1.0$, so that:

$$1.0 = \sum_{n=1}^{\infty} a_n \left[e^{-\lambda_n L} + \frac{\lambda_n + Bi_L}{\lambda_n - Bi_L} e^{\lambda_n L} \right] J_0(\lambda_n R).$$

Therefore,

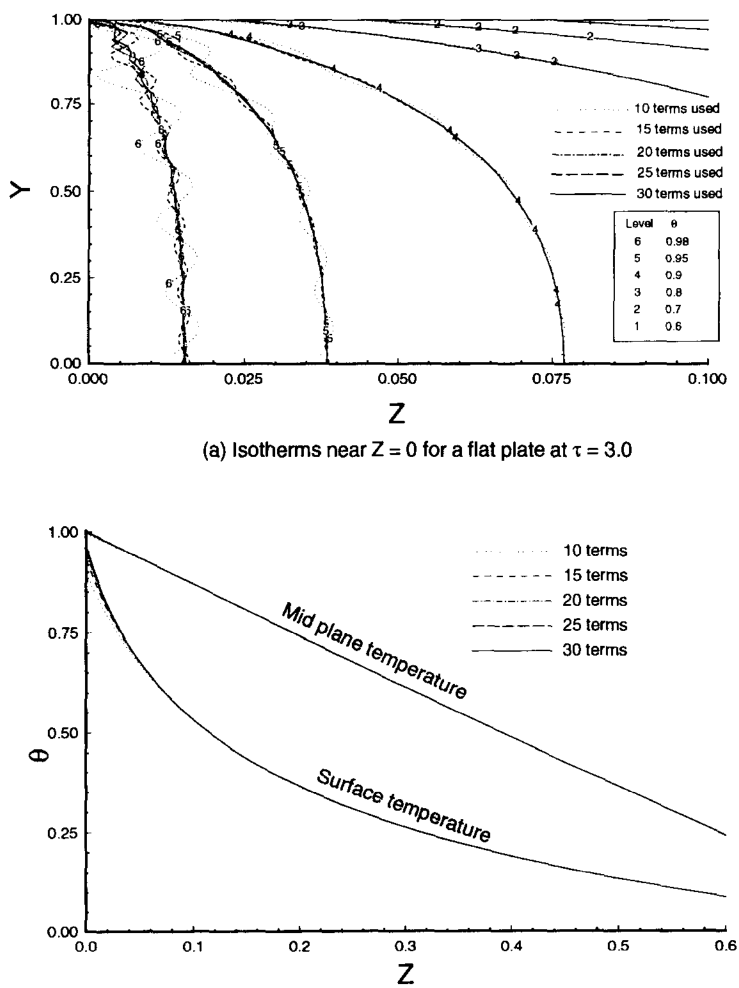
$$a_n \left[e^{-\lambda_n L} + \frac{\lambda_n + Bi_L}{\lambda_n - Bi_L} e^{\lambda_n L} \right] = \frac{\int_0^1 R J_0(\lambda_n R)(1) \mathrm{d}R}{\int_0^1 R J_0^2(\lambda_n R) \mathrm{d}R} = \frac{2Bi}{(\lambda_n^2 + Bi^2) J_0(\lambda_n)}.$$

This gives the following complete solution for ϕ as:

$$\phi(\xi,R) = 2 \sum_{n=1}^{\infty} \frac{Bi}{(\lambda_n^2 + Bi^2) J_0(\lambda_n)} \left[\frac{e^{\lambda_n \xi} + \left(\frac{\lambda_n + Bi_L}{\lambda_n - Bi_L} \right) e^{-\lambda_n \xi}}{e^{-\lambda_n L} + \left(\frac{\lambda_n + Bi_L}{\lambda_n - Bi_L} \right) e^{\lambda_n L}} \right] J_0(\lambda_n R) \quad (19)$$

where the values of λ_n [$n = 1,2,3,\ldots$] are given by the roots of the equation (16).

Now equation (10) is solved with the boundary conditions given by equation (11). Again, a product solution of the form $\psi(\xi,R,t) = Z_2(\xi) R_2(R) T_1(t)$ is employed and when this is introduced in equation (10), we obtain the following ODEs:



(a) Isotherms near Z = 0 for a flat plate at τ = 3.0

(b) Surface and mid-plane temperature for τ = 3.0 for a flat plate

FIG. 2. (a) Isotherms and (b) temperature distributions in a flat plate, using a finite number of terms in the analytical solution for $Pe = 0.2$, $Bi = Bi_L = 5.0$.

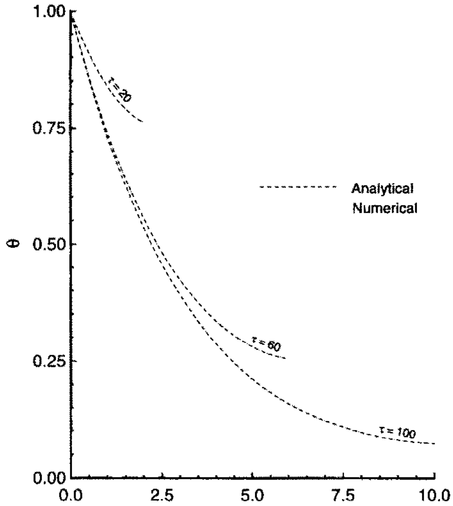


FIG. 3. Comparison between analytical and numerical results for the transient temperature distribution in a one-dimensional moving cylindrical rod at $Pe = 0.2$, with $Bi = Bi_L = 0.1$.

$$\frac{d^2Z_2}{d\xi^2}+\mu^2 Z_2 = 0 \tag{20}$$

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dR_2}{dR}\right)+v^2R_2 = 0 \tag{21}$$

$$\frac{d\tau_1}{dt}+(\mu^2+v^2)T_1 = 0 \tag{22}$$

with the following conditions:

$$\frac{dR_2}{dR}(0) = 0.0; \quad \frac{dR_2}{dR}(1) = -BiR_2(1);$$

$$\frac{dZ_2}{d\xi}(0) = -Bi_L Z_2(0); \quad Z_2(-L) = 0. \tag{23}$$

The solution to equation (20) is

$$Z_{2m}(\xi) = A_m\cos(\mu_m\xi)+B_m\sin(\mu_m\xi).$$

B_m can be easily eliminated by the use of the boundary conditions given by equation (23). Thus,

$$Z_{2m}(\xi) = \frac{A_m}{\sin(\mu_m L)}\sin\mu_m(\xi+L)$$

where the roots μ_m are to be obtained from

$$\mu_m\cot(\mu_m L) = -Bi_L. \tag{24}$$

Also since, equation (21) is of the same form as equation (12) and since both the equations have similar boundary conditions, the solution to equation (21) is of the same form as that to equation (12), i.e.

$$R_{2k}(R) = B_k J_0(v_k R) \tag{25}$$

where v_k is given by

$$v_k J_1(v_k)-BiJ_0(v_k) = 0. \tag{26}$$

The general solution to equation (22) is

$$T_{mk}(t) = C_{mk}\,e^{-(\mu_m^2+v_k^2)t}. \tag{27}$$

Therefore, the product solution can be written as:

$$\psi(\xi,R,t) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk}\,e^{-(\mu_m^2+v_k^2)t} \times \frac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}J_0(v_k R) \tag{28}$$

where $a_{mk} = A_m B_k C_{mk}$. Using the initial conditions from equation (11), we get

$$1-\phi(\xi,R) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk}\frac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}J_0(v_k R).$$

This is a double Fourier series from which the solution can be obtained as:

$$a_{mk} = \frac{\displaystyle\int_0^1\int_{-L}^0 RJ_0(v_k R)[1-\phi(\xi,R)]\frac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}\,dR\,d\xi}{\displaystyle\int_0^1\int_{-L}^0 RJ_0^2(v_k R)\frac{\sin^2\mu_m(\xi+L)}{\sin^2(\mu_m L)}\,dR\,d\xi}. \tag{29}$$

This equation is difficult to evaluate analytically, but can easily be determined by numerical integration. Thus the solution $\theta(\xi,R,t)$ is obtained by combining equations (19) and (28) and can be written as

$$\theta(\xi,R,t) = 2\sum_{n=1}^{\infty}\frac{Bi}{(\lambda_n^2+Bi^2)J_0(\lambda_n)} \times \left[\frac{e^{\lambda_n\xi}+\left(\dfrac{\lambda_n+Bi_L}{\lambda_n-Bi_L}\right)e^{-\lambda_n\xi}}{e^{-\lambda_n L}+\left(\dfrac{\lambda_n+Bi_L}{\lambda_n-Bi_L}\right)e^{\lambda_n L}}\right]J_0(\lambda_n R) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk}\,e^{-(\mu_m^2+v_k^2)t}\frac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}J_0(v_k R) \tag{30}$$

where λ_n, μ_m and v_k are the roots of equations (16), (24) and (26), respectively, while a_{mk} is obtained from equation (29).

2.2. *Flat plate*

The derivation of the temperature distribution for a moving flat plate of large width is very similar to that for a moving cylinder. The governing equation for the 2D transport in the case of a wide flat plate is

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial \tau'}+U_s\frac{\partial T}{\partial z}\right) = \frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial y^2}. \tag{31}$$

Using the same non-dimensionalization as before, with $Y = y/R_0$, where R_0 now is the half-width of the flat plate, and employing the same co-ordinate transformation as in (4) the equation becomes:

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial Y^2} \tag{32}$$

with the same boundary conditions as those given by

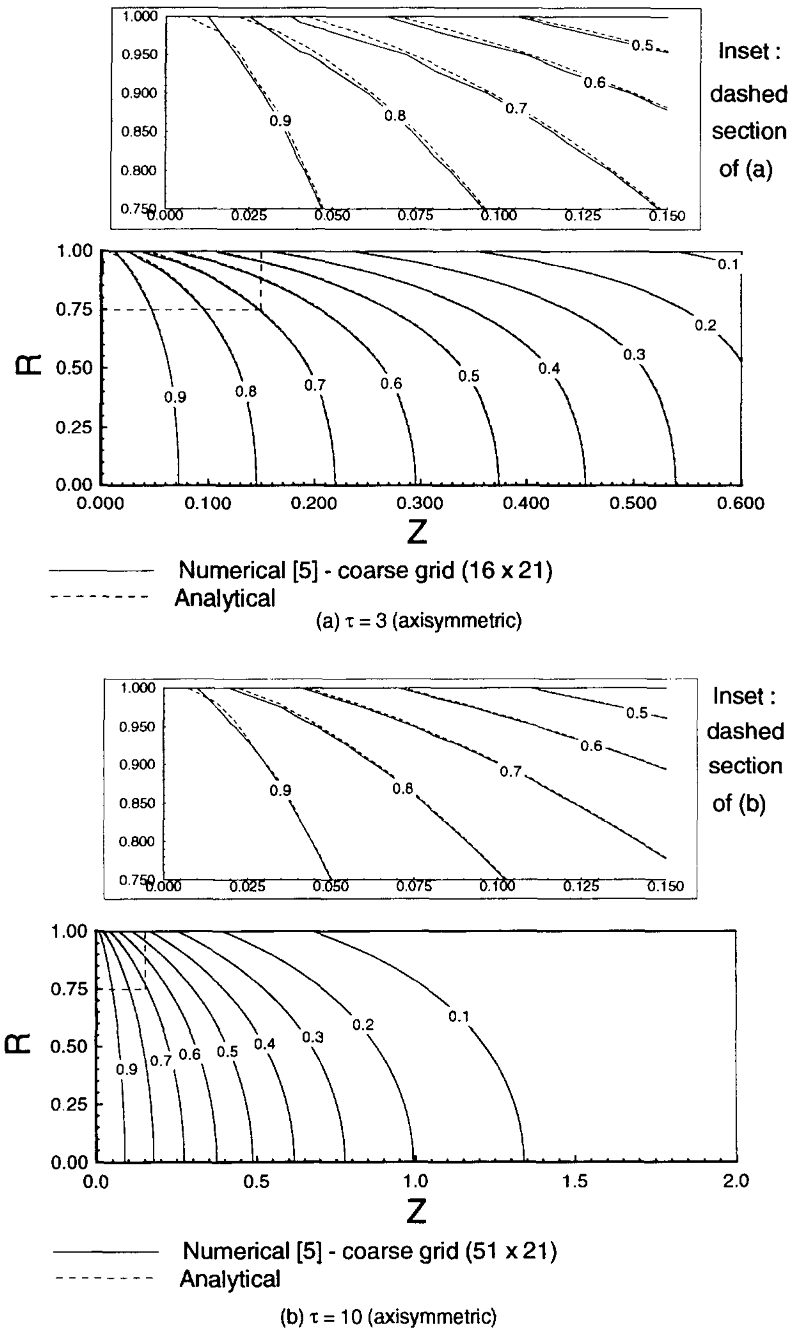


Fig. 4. Comparison between analytical and numerical results for the transient isotherms (using a coarse grid for the numerical case) in a cylinder moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

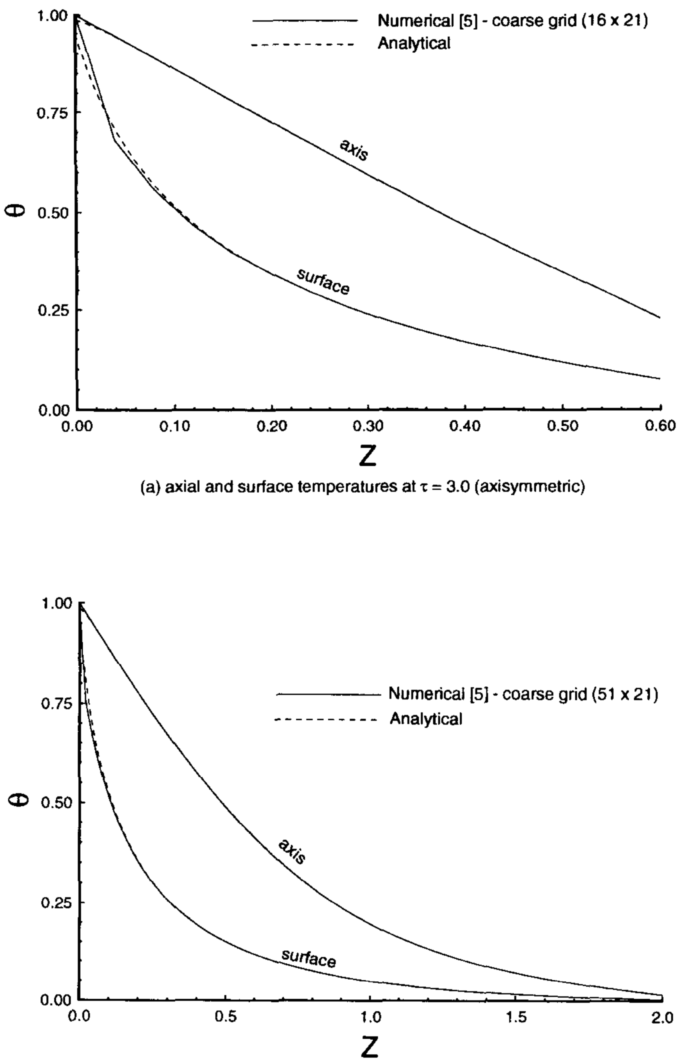
equation (6). Again the solution is obtained by the superposition of the 'pseudo-steady' solution $\phi(\xi, Y)$ and $\Psi(\xi, Y, t)$, where the equations and boundary conditions are similar to equations (8), (10), (9) and (11). Again a product solution for $\phi(\xi, Y)$ is sought in the form $Z_1(\xi)Y_1(Y)$. This, along with the boundary conditions, gives the following solution for $\phi(\xi, Y)$:

$$\phi(\xi, Y) = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n}{(\lambda_n + \sin\lambda_n\cos\lambda_n)} \times \left[\frac{e^{\lambda_n\xi} + \left(\dfrac{\lambda_n + Bi_L}{\lambda_n - Bi_L}\right)e^{-\lambda_n\xi}}{e^{-\lambda_n L} + \left(\dfrac{\lambda_n + Bi_L}{\lambda_n - Bi_L}\right)e^{\lambda_n L}}\right]\cos(\lambda_n Y) \quad (33)$$

where λ_n now is given by the following equation:

$$\lambda_n \tan(\lambda_n) = Bi. \quad (34)$$

Using this solution for ϕ, the equation for ψ is solved



(a) axial and surface temperatures at $\tau = 3.0$ (axisymmetric)

(b) axial and surface temperatures at $\tau = 10.0$ (axisymmetric)

FIG. 5. Comparison between analytical and numerical results for the transient centerline and surface temperature distributions (using a coarse grid for the numerical case) for a cylinder moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

as a product solution $Z_2(\xi)Y_2(Y)T_1(t)$, resulting in the following equation:

$$\psi(\xi,Y,t) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk}\,e^{-(\mu_m^2+\nu_k^2)t} \times \frac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}\cos(\nu_k Y) \quad (35)$$

where μ_m is the solution of the equation (24), while ν_k is obtained from the equation:

$$\nu_k \tan(\nu_k) = Bi. \quad (36)$$

Using the same procedure as before for a double Fourier series, the expression for a_{mk} is obtained as:

$$a_{mk} = \frac{\int_0^1\int_{-L}^0 \cos(\nu_k Y)[1-\phi(\xi,Y)]\dfrac{\sin\mu_m(\xi+L)}{\sin(\mu_m L)}\,\mathrm{d}Y\,\mathrm{d}\xi}{\int_0^1\int_{-L}^0 \cos^2(\nu_k Y)\dfrac{\sin^2\mu_m(\xi+L)}{\sin^2(\mu_m L)}\,\mathrm{d}Y\,\mathrm{d}\xi}. \quad (37)$$

Therefore, finally putting everything together, the complete solution for the time-dependent temperature distribution in a moving (wide) flat plate of finite length can be written as:

$$\theta(\xi,Y,t) = 2\sum_{n=1}^{\infty}\frac{\sin\lambda_n}{(\lambda_n+\sin\lambda_n\cos\lambda_n)}$$

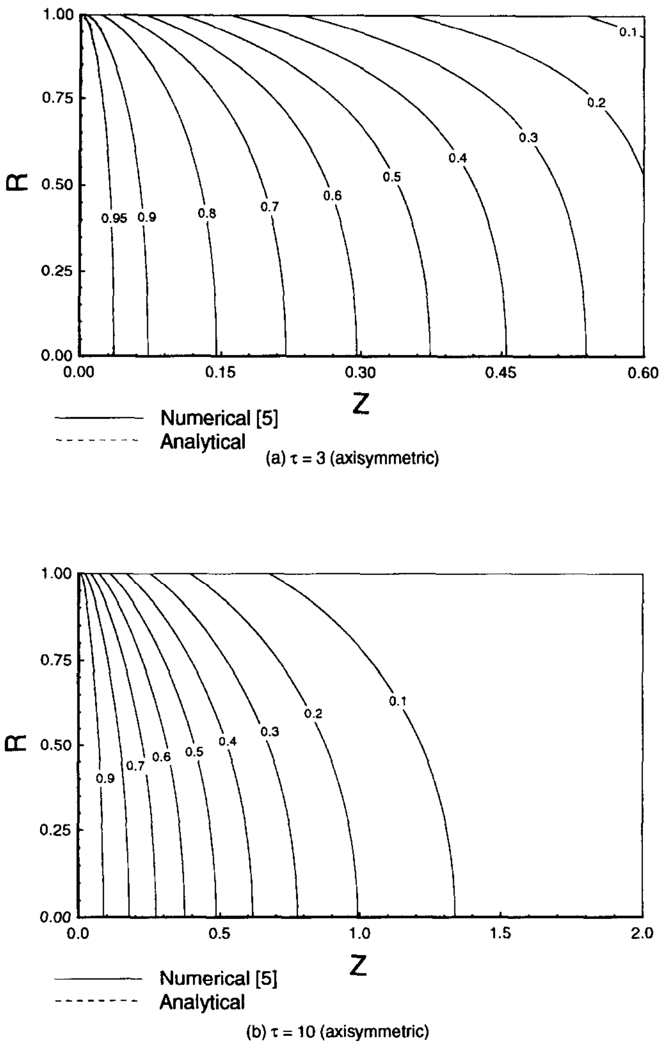


Fig. 6. Comparison between analytical and numerical results for the transient isotherms (using a fine grid for the numerical case) in a cylinder moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

$$\times \left[\frac{e^{\lambda_n \xi} + \left(\dfrac{\lambda_n + Bi_L}{\lambda_n - Bi_L}\right) e^{-\lambda_n \xi}}{e^{-\lambda_n L} + \left(\dfrac{\lambda_n + Bi_L}{\lambda_n - Bi_L}\right) e^{\lambda_n L}}\right] \cos(\lambda_n Y)$$

$$+ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} a_{mk}\, e^{-(\mu_m^2 + v_k^2)t} \frac{\sin \mu_m(\xi + L)}{\sin(\mu_m L)} \cos(v_k Y) \tag{38}$$

where λ_n, μ_m and v_k are the roots of equations (34), (24) and (36), respectively, while a_{mk} is obtained from equation (37).

3. ANALYTICAL SOLUTION FOR THE 1D TRANSIENT PROBLEM

When the Biot number Bi is small, radial temperature uniformity can be assumed. The non-dimensional governing equations for a cylinder with radial lumping is obtained as:

$$\frac{\partial \theta}{\partial \tau} + Pe \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Z^2} - 2Bi\theta. \tag{39}$$

Following the transformations given by equation (4), the equation (39) reduces to

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \xi^2} - 2Bi\theta \tag{40}$$

with the initial and boundary conditions given by

$$\theta(\xi, 0) = 1.0; \quad \theta(-L, t) = 1.0$$

and

$$\frac{\partial \theta}{\partial \xi}(0, t) = -Bi_L \theta(0, t). \tag{41}$$

Since, the boundary conditions are non-homo-

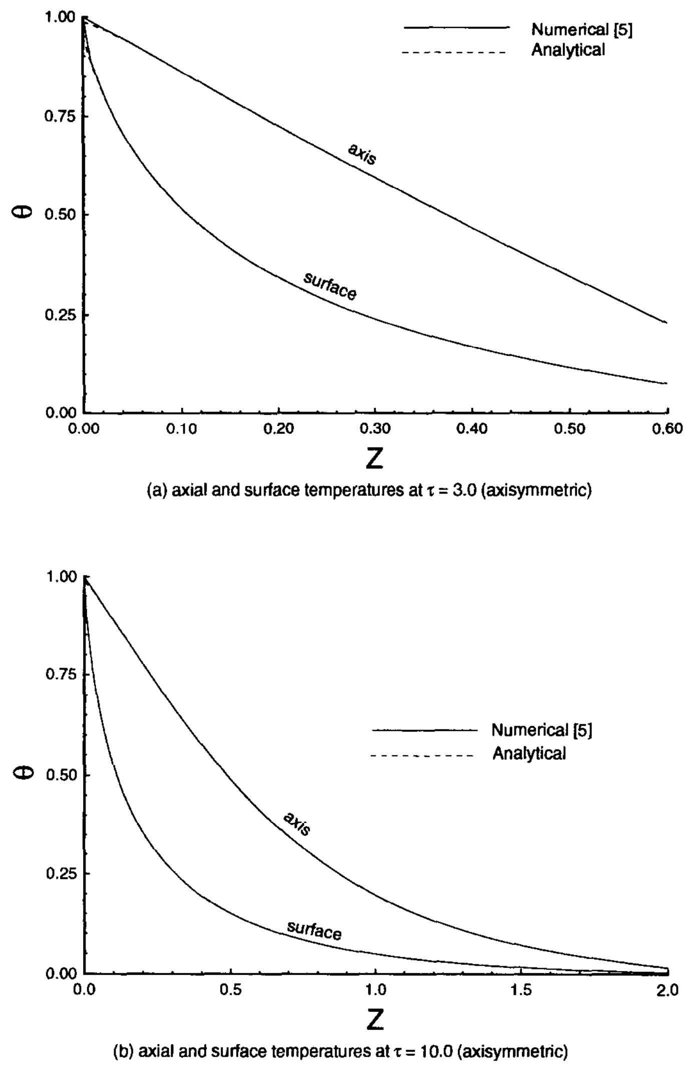


FIG. 7. Comparison between analytical and numerical results for the transient centerline and surface temperature distributions using a fine grid for the numerical case, for a cylinder moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

geneous, the principle of superposition has to be used. Noting that the problem has a 'pseudo-steady' solution as $t \to \infty$, the following superposition is employed:

$$\theta(\xi,t) = \psi(\xi,t)+\phi(\xi). \tag{42}$$

Here, $\phi(\xi)$ gives the solution for the 'pseudo-steady' part of the problem

$$\frac{d^2\phi}{d\xi^2} - 2Bi\phi = 0 \tag{43}$$

with the given boundary conditions

$$\phi(-L) = 1 \quad \text{and} \quad \frac{d\phi}{d\xi}(0) = -Bi_L\phi(0). \tag{44}$$

This is an ordinary differential equation, for which the following solution is obtained:

$$\phi(\xi) = \frac{\left[e^{\sqrt{(2Bi)}\xi} + \left(\dfrac{Bi_L+\sqrt{(2Bi)}}{-Bi_L+\sqrt{(2Bi)}}\right)e^{-\sqrt{(2Bi)}\xi}\right]}{\left[e^{-\sqrt{(2Bi)}L} + \left(\dfrac{Bi_L+\sqrt{(2Bi)}}{-Bi_L+\sqrt{(2Bi)}}\right)e^{\sqrt{(2Bi)}L}\right]}. \tag{45}$$

Also, $\psi(\xi,t)$ is given by

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial\xi^2} - 2Bi\psi \tag{46}$$

with the following initial and boundary conditions:

$$\psi(\xi,0) = 1.0-\phi(\xi); \quad \psi(-L,t) = 0.0$$

and

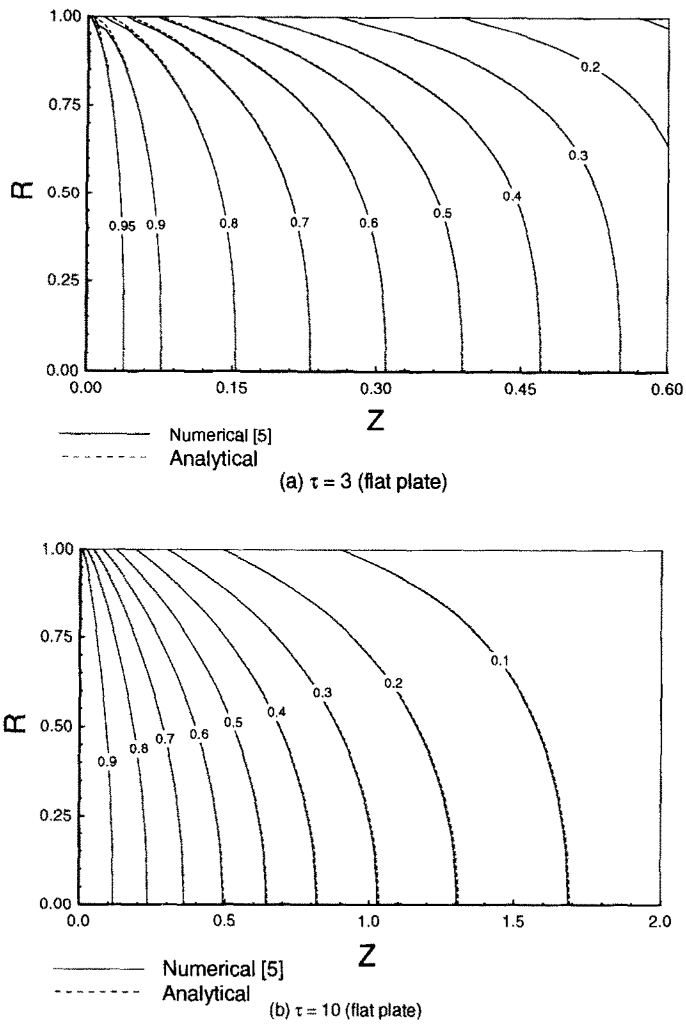


FIG. 8. Comparison between analytical and numerical results for the transient isotherms in a flat plate moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

$$\frac{\partial \psi}{\partial \xi}(0,t) = -Bi_L \psi(0,t) \tag{47}$$

so that, when the governing equations and the boundary conditions for $\psi(\xi,t)$ and $\phi(\xi)$ are added, they yield the equation for $\theta(\xi,\tau)$. Equation (46), with the given boundary conditions, equation (47), is not admissible to a solution by the method of separation of variables. To make the solution feasible, the following transformation is employed

$$\psi(\xi,t) = u(\xi,t)e^{-2Bit}. \tag{48}$$

With this transformation, equations (46) and (47) reduce to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} \tag{49}$$

with the following initial and boundary conditions:

$$u(\xi,0) = 1.0-\phi(\xi); \quad u(-L,t) = 0.0$$

and

$$\frac{\partial u}{\partial \xi}(0,t) = -Bi_L u(0,t). \tag{50}$$

For solving equation (49), separation of variables is used, with $u(\xi,t) = X_1(\xi)\tau_1(t)$, which when substituted in equation (49) gives

$$\frac{d^2X_1}{d\xi^2} + \lambda^2 X_1 = 0 \tag{51}$$

$$\frac{d\tau_1}{dt} = -\lambda^2\tau_1. \tag{52}$$

The solution to equation (51), with the boundary conditions given by equation (50), can be written as

$$X_{1m}(\xi) = A_m \sin\lambda_m(\xi+L) \tag{53}$$

where λ_m are the roots of the equation

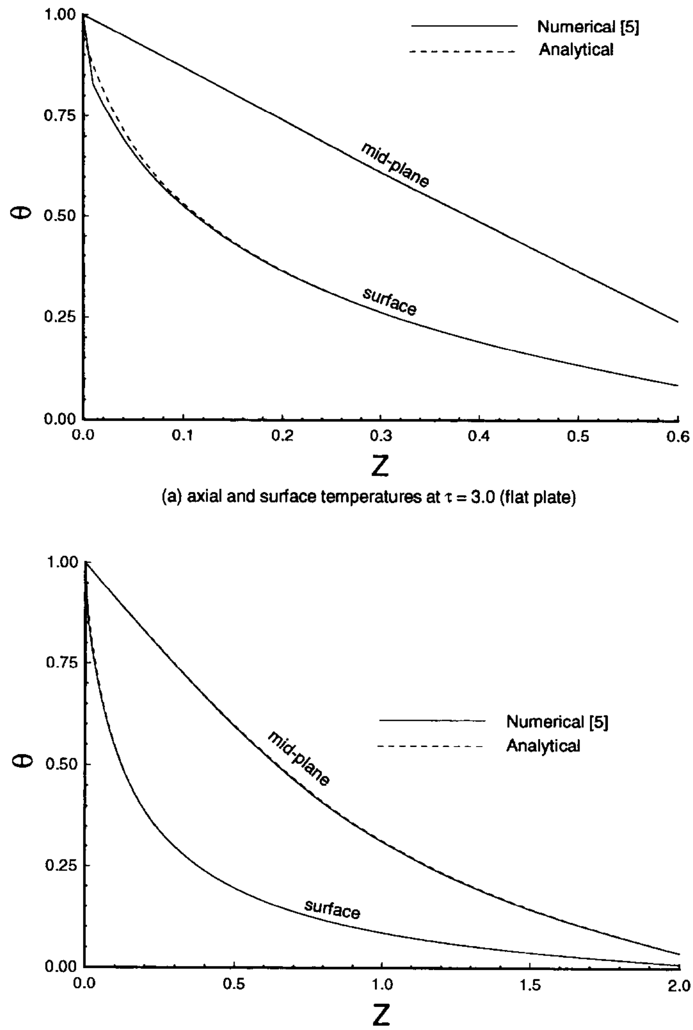


FIG. 9. Comparison between analytical and numerical results for the transient midplane and surface temperature distributions for a flat plate moving at $Pe = 0.2$, with $Bi = Bi_L = 5.0$ at (a) $\tau = 3.0$ and (b) $\tau = 10.0$.

$$\lambda_m \cot(\lambda_m L) + Bi_L = 0. \quad (54)$$

The general solution for equation (52) is

$$\tau_1(\tau) = C_m \, e^{-\lambda_m^2 t}. \quad (55)$$

Therefore, using $u = X_1 \tau_1$, the general solution for $u(\xi,t)$ is:

$$u(\xi,t) = \sum_{m=0}^{\infty} B_m \sin \lambda_m(\xi+L)\, e^{-\lambda_m^2 t} \quad (56)$$

where $B_m = A_m C_m$. The coefficient B_m can be obtained from the initial condition. Since $u(\xi,0) = 1.0-\phi(\xi)$, B_m can be written, using the Fourier series properties, as the following integral:

$$B_m = \frac{\int_{-L}^{0} [1.0-\phi(\xi)] \sin \lambda_m(\xi+L)\, d\xi}{\int_{-L}^{0} \sin^2 \lambda_m(\xi+L)\, d\xi}. \quad (57)$$

The analytical expression for B_m can be determined by carrying out the integration in the equation (57). However, the resulting expression is lengthy and cumbersome. So the integration is done numerically, to finally give the solution $\psi(\xi,t)$ as

$$\psi(\xi,t) = \sum_{m=0}^{\infty} B_m \sin \lambda_m(\xi+L)\, e^{-(\lambda_m^2+2Bi)t}. \quad (58)$$

Therefore, using equation (42), the temperature distribution $\theta(\xi,t)$ can be written as

$$\theta(\xi,t) = \frac{\left[e^{\sqrt{(2Bi)}\xi} + \left(\frac{Bi_L+\sqrt{(2Bi)}}{-Bi_L+\sqrt{(2Bi)}} \right) e^{-\sqrt{(2Bi)}\xi} \right]}{\left[e^{-\sqrt{(2Bi)}L} + \left(\frac{Bi_L+\sqrt{(2Bi)}}{-Bi_L+\sqrt{(2Bi)}} \right) e^{\sqrt{(2Bi)}L} \right]} + \sum_{m=0}^{\infty} B_m \sin \lambda_m(\xi+L)\, e^{-(\lambda_m^2+2Bi)t} \quad (59)$$

where λ_m and B_m are given by equations (54) and (57), respectively.

4. RESULTS AND DISCUSSION

Some of the transient results obtained from the analytical solutions are compared with those obtained numerically by Jaluria and Singh [5]. In the numerical scheme, the length L was taken as unchanged over a finite time increment $\Delta\tau$ and was then taken at the increased value for the next time step, as shown in Fig. 1. The Gauss–Seidel iterative procedure was employed, in conjunction with the Crank–Nicolson scheme, to obtain the temperature distribution at each time step. For the analytical solution, numerical integration was carried out for finding the coefficients a_{mk} and B_m for the two-dimensional and one-dimensional cases, respectively. Romberg integration is used to determine these coefficients [6]. Also, to find the roots λ_n, μ_m and ν_k for the two-dimensional cases, and λ_m for the one-dimensional case, the Newton–Raphson root solving method is used [6]. Since the analytical solution is the summation of an infinite series, only a finite number of terms are taken for obtaining the analytical solution and the effect of the number of terms on the analytical solution is determined.

It is found that the solution of the 'pseudo-steady' circumstance is very close to the actual solution. Thus, the effect of the 'pseudo-transient' terms, which are $\Psi(\xi,R,t)$ or $\Psi(\xi,Y,t)$ for the two-dimensional cases and $\Psi(\xi,t)$ for the one-dimensional case, is relatively small for $\tau > 0.2$, which is considered for most of the results presented here. So unless the solution is needed for very small time, that is when τ is less than 0.2, a 'pseudo-steady' approximation can be used, to obtain the analytical solution more easily. This also indicates why the numerical solutions that assume a quasi-steady behavior yield accurate results at large time.

The isotherms and the surface and mid-plane temperatures for the two-dimensional flat plate case, taking a finite number of terms for the analytical solution, are shown in Fig. 2. The main difference in the results, due to a different number of terms being included in the solution, is found near the slot from where the material emerges, especially near the surface. It is found that, when the number of terms used is more than 25, the solution converges, that is the solution does not change appreciably when an additional number of terms is used. For instance, when 30 terms are used, the difference from the results obtained with 25 terms is less than 0.1%. Therefore, for all the results obtained from the analytical solution, the number of terms is taken as 25. From a similar study of the axisymmetric problem, it was found that the number of terms required to get a converged solution is also 25 in that case. For the one-dimensional case, the 'pseudo-steady part', $\phi(\xi)$, does not consist of a series solution, only the 'pseudo-transient' part, $\psi(\xi,t)$, does. Again, except for very small times, the contribution of the latter term is small, and even then the inclusion of the first 20 terms was found to be sufficient for a converged, accurate, solution.

The comparison of the time-dependent temperature distributions in a cylinder for the one-dimensional transient case, at $Bi = Bi_L = 0.1$ and $Pe = 0.2$, with the numerical results obtained by Jaluria and Singh [5] is shown in Fig. 3. An excellent agreement is obtained, lending support to the numerical scheme used.

For the axisymmetric case, the transient isotherms are obtained from the analytical solution, and are compared with the numerical results. When a relatively coarse grid (51×21 for $\tau = 10$ and 16×21, for $\tau = 3$) is used, the comparisons of the isotherms (Fig. 4) as well as of the surface and axial temperature distributions (Fig. 5), show only a slight discrepancy in the surface temperature near the slot from where the material emerges. For a finer grid (251×101 grid and 76×101 for the aforementioned cases, respectively) an extremely close agreement is observed, as seen from Fig. 6. Also the transient axial and surface temperatures are shown in Fig. 7, showing excellent agreement. For a coarser grid, the difference between the analytical and numerical results was found to increase.

Another important observation can be made from Fig. 4. It is observed from the insets in Figs. 4(a) and (b), showing the results at small Z, that the discrepancy in temperature near the slot is considerably more for $\tau = 3$ as compared to that for $\tau = 10$. As mentioned earlier, the numerical solution assumes the length of the cylinder to remain unchanged over a time step, $\Delta\tau$, and, therefore, at small time, unless the grid is considerably more fine, the numerical solution is not very accurate. The error, indicated in the inset of Fig. 4(a) is even more severe for smaller times. When a finer grid is used, however, as indicated in Fig. 6, the discrepancy between the numerical solution and the analytical solution diminishes, even though it is still higher at smaller times. In principle, at small time the numerical solution can be obtained accurately if a large number of grid points are used. However, for very small time, and consequently very fine grid, this might introduce considerable amount of roundoff error, in addition to consuming large amount of CPU time. Therefore, it can be concluded from the previous discussion that the analytical method is a useful tool to check the accuracy of the numerical scheme and to provide results at short times.

The transient isotherms for the flat plate are shown in Fig. 8, again showing very close agreement with the numerical solution using fine grid (151×41 grid for $\tau = 10$ and 46×41 grid for $\tau = 3$). The transient mid-plane and surface temperatures are shown in Fig. 9. Again, except for the surface temperature near the slot, the agreement between the analytical and the numerical results is excellent.

For all the cases considered here, the analytical solution can be used to compute the temperature distributions at small time accurately, but, because of the assumption of constant length over a time increment,

$\Delta\tau$, the numerical solution for small time is somewhat in error, as discussed before.

Even though the analytical solution obtained here is a compact expression, it also requires some computational effort to determine the roots λ_m, v_k, μ_m, etc. Also, numerical integration is needed to determine the coefficients a_{mk}, B_m, etc. On the other hand, the numerical solution is much easier to implement, even though the CPU time required to carry out the necessary numerical calculations required for the analytical solution for the two-dimensional cases is at least 20 times less than that required for the numerical solution.

The analytical solution is possible only because of the assumption of constant heat transfer coefficient at the surface and at the end of the moving cylinder/plate. In real circumstances, these coefficients are not known. Therefore, in general, a complete numerical study considering the conjugate transport in the fluid and in the solid must be carried out to solve the full, elliptic, problem. This cannot be done analytically, unless some severe (and unreal) assumptions are made. It must also be mentioned that, even though there are a number of numerical solutions available for an infinite moving plate [9, 10] or cylinder [11], with conjugate fluid transport from its surface, there are no numerical results available for the case presented here, involving a finite length and a moving boundary, that includes conjugate fluid transport. However, the analytical solution presented here may be useful in developing a suitable numerical scheme for the conjugate problem.

5. CONCLUSIONS

An analytical solution is obtained for the one- and two-dimensional transient temperature distributions in a finite-length moving cylinder/plate subjected to a known heat transfer coefficient at the surface. The results are obtained in terms of a series solution, involving integrals. A root solving method is used to obtain the roots required in the series solution, while numerical integration is used to determine the coefficients. It has been found that a finite number of terms can be used for obtaining the analytical solution, with reasonable accuracy. In particular, 25 terms for the two-dimensional problems were found to be sufficient. For the one-dimensional problem, the 'pseudo-transient' part, which consists of a series solution, is almost negligible at large time, while, for small time, employing 20 terms for the 'pseudo-transient' part of the solution was found to be sufficient. Also, the analytical solution, derived as a combination of 'pseudo-steady' and 'pseudo-transient' parts, was found to be dominated by the former, with the latter part being almost negligible. This indicates why the numerical solutions that assume a quasi-steady behavior yield accurate results at large time. A comparison between the analytical results and the numerical results obtained earlier showed excellent agreement. It is concluded that the numerical solution is much easier to implement. Also, in the case of a real life problem, when the heat transfer coefficient at the surface or the end of the material is not known, the full, conjugate problem must be solved numerically for both the fluid and the solid to obtain the temperature distribution in the moving material. However, the analytical solution is valuable, because it provides a means of validating the numerical schemes for similar problems. Also, the analytical solution can be used to obtain an accurate temperature distribution near the slot, especially at small time, which the numerical solutions generally fail to provide.

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